

Functional Integral approach to quasi-particles in Fermi liquid theory

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Abstract

In this paper we propose a new way of organizing many-body perturbation theory in the Path-integral formulation where a set of quasi-particle wavefunctionals ψ 's are introduced and are identified with quasi-particles in Landau Fermi liquid theory. We show how Fermi liquid theory can be obtained through ψ 's in this new framework of perturbation theory, where the only assumption is adiabaticity between non-interacting and interacting states and the quasi-particle renormalization factor z does not appear explicitly. Consequences of our new formulation are discussed.

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For four decades Fermi liquid theory has formed the basis for our understanding of interacting fermion systems like electron liquids in metals and normal state of liquid He^3 . The theory was originally proposed phenomenologically by Landau [1,2] and was later put on a firm basis by a formal 'proof' using diagrammatic perturbation theory techniques [3–5]. Although formally exact, the existing perturbation theory formulation of Fermi liquid theory has a drawback. In the original Fermi liquid theory of Landau, the only assumption was adiabaticity or a one-to-one correspondence between non-interacting states (bare particles) and states when interaction is turned on (quasi-particles). However, in the existing diagrammatic technique, perturbation theory was carried out in terms of *bare particles* and the connection between bare particles and quasi-particles is established by assuming that there is a non-vanishing overlap $z^{1/2} \sim O(1)$ between the two. It would be much more satisfying if the perturbation theory can be formulated directly in terms of quasi-particles where the introduction of renormalization factor z can be avoided, as is in the original Landau formulation.

In this paper we shall show that within the path-integral formulation it is possible to formulate perturbation theory in terms of a set of quasi-particle wave-functionals which can be identified with quasi-particles in Fermi liquid theory. With these wave-functionals Fermi liquid theory can be derived directly without referring to the quasi-particle renormalization factor z . The one-particle Green's function will be studied where we shall show that the quasi-particle renormalization factor $z^{1/2}$ measures the overlap between bare-particle states and our quasi-particle states on the Fermi surface. For simplicity we shall consider a gas of spinless fermions interacting through a scalar potential $v(r)$. The formulation can be generalized to include spin easily. We shall restrict us to zero temperature where the perturbation theory is formulated in terms of real-time Path Integral and Green's functions. The Hamiltonian is, in momentum space,

$$H = \sum_{\vec{k}} \epsilon(\vec{k}) f_{\vec{k}}^+ f_{\vec{k}} + \lim_{\eta \rightarrow 0} e^{-\eta|t|} \frac{1}{2} \sum_{\vec{q}} v(q) \rho(\vec{q}) \rho(-\vec{q}), \quad (1)$$

where $\epsilon(\vec{k}) = (\hbar \vec{k})^2 / 2m$ and $f(f^+)_{\vec{k}}$'s are fermion annihilation(creation) operators. $\rho(\vec{q}) =$

$\sum_{\vec{k}} f_{\vec{k}+\vec{q}}^+ f_{\vec{k}}$ is the density operator for fermions. Notice that we have inserted a $e^{-\eta|t|}$ factor in front of the interaction term to emphasize that the states in the interacting system are obtained from states of the non-interacting system by turning on the interaction *adiabatically*, as is required in Landau Fermi liquid theory. In particular, the time evolution operator $U(t, t')$, when operate on the ground state of the non-interacting system at $t' = -\infty$, turns the state into the ground state of the interacting system at time t . U can be expressed in terms of a path integral over (fermionic) coherent states Ψ, Ψ^+ [6]. In particular,

$$\lim_{T \rightarrow \infty} \text{Tr} [U(T, -T)] = \int D\Psi D\Psi^+ \int D\phi e^{\frac{i}{2} \int_{-T}^T dt' \int d^d q \frac{\phi(\vec{q}, t') \phi(-\vec{q}, t')}{v(q)}} e^{\frac{i}{\hbar} \int_{-T}^T L(\phi; \Psi^+, \Psi, t') dt'}, \quad (2a)$$

where

$$L(\phi; \Psi^+, \Psi, t) = \int d^d x \left\{ i\hbar \Psi^+(\vec{x}, t) \frac{\partial \Psi(\vec{x}, t)}{\partial t} - \frac{\hbar^2}{2m} |\nabla \Psi(\vec{x}, t)|^2 - \lim_{\eta \rightarrow 0} e^{-\frac{\eta}{2}|t|} \phi(\vec{x}, t) \Psi^+(\vec{x}, t) \Psi(\vec{x}, t) \right\}, \quad (2b)$$

is the Lagrangian for a system of *non-interacting* fermions moving in the auxillary field $\phi(\vec{x}, t)$, introduced through an Hubbard-Stratonovich transformation from the interaction term in the original Hamiltonian. In this form, the time evolution operator U can be interpreted as a weighted sum of time evolution operators $U(\phi)$'s, where $U(\phi)$ is the time evolution operator for system of non-interacting fermions moving in given auxillary field $\phi(\vec{x}, t)$, with each configuration of $U(\phi)$ weighted by the factor $e^{i \int d^d q \int dt \frac{\phi(\vec{q}) \phi(-\vec{q})}{2v(q)}}$ in the sum [6]. The interacting ground state wavefunction can thus be interpreted as superposition of 'ground state' wavefunctions $|G(\phi) \rangle$'s, generated by operators $U(\phi)$'s acting on the free fermion ground state at $t' = -\infty$, and weighted by the same $e^{i \int \frac{\phi^2}{2v}}$ factors. Notice also that the auxillary field ϕ is turned on adiabatically at $t = -\infty$, as a result of adiabaticity of the original interacting fermion problem.

Next we introduce the Green's function operator $G[\phi]$, which is a functional of ϕ field, with

$$G^{-1}[\phi] = i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - \lim_{\eta \rightarrow 0} e^{-\frac{\eta}{2}|t|} \phi(\vec{x}, t), \quad (3a)$$

and with Fourier transform

$$G_{k,k'}^{-1}[\phi] = (\hbar\omega - \epsilon(\vec{k}) + i\delta_{\vec{k}})\delta_{k,k'} - \phi(k - k'), \quad (3b)$$

where $k = (\vec{k}, \omega)$ and $\delta_{\vec{k}} = \delta \text{sgn}(\xi_{\vec{k}})$, where $\xi_{\vec{k}} = \epsilon(\vec{k}) - \mu$ and μ = chemical potential. Notice that the adiabaticity requirement is handled by introducing the $i\delta_{\vec{k}}$ term in G as in usual perturbation theory. The ground state energy, one particle Green's function, etc, can be determined as appropriate functional averages of G or functions of G over ϕ field [6]. $G_{k,k'}[\phi]$ can be expanded directly in a power series of ϕ as in usual perturbation theory. However, here we shall rearrange the perturbation expansion in a slightly different way, by introducing first the *eigenfunctions* and corresponding *eigenvalues* of $G^{-1}[\phi]$, where $G^{-1}[\phi]\psi_k[\phi] = \lambda_k[\phi]\psi_k[\phi]$. Notice that both ψ_k 's and λ_k 's are functionals of ϕ . The perturbative eigenstates ψ_k satisfies the Lippman-Schwinger equation

$$\psi_k(x; [\phi]) = A_k[\phi]\psi_k^{(0)}(x) + \int d^{d+1}x G_0(\lambda_k, x - x')\phi(x')\psi_k(x'; [\phi]), \quad (4)$$

where $x = (\vec{x}, t)$, A_k is a renormalization factor determined by $\langle \psi_k | \psi_k \rangle = 1$ and $G_0(\lambda, x)$ is the Fourier transform of the unperturbed Green's function, $G_0(\lambda, k') = (\lambda - (\omega' - \epsilon(\vec{k}') + i\delta_{\vec{k}}))^{-1}$. $\psi_k^{(0)}(x) \sim e^{-i(\omega t - \vec{k} \cdot \vec{x})}$ is an unperturbed eigenstate of G_0^{-1} . The eigenvalue λ_k can be written as $\lambda_k = \omega - \epsilon(\vec{k}) - \Sigma(k; [\phi]) + i\delta_{\vec{k}}$, where $\Sigma(k; [\phi]) = \int_0^1 dg \int d^{d+1}x \phi(x) |\psi_k(x; [g\phi])|^2$, by the Hellmann-Feynman theorem. $\int_0^1 dg$ is a coupling constant integral. Using the fact that $G^{-1}[\phi]$ is a first order differential equation in time, it can be shown easily that $\psi_{\vec{k}, \omega + \Omega}(x; [\phi]) = e^{-i\Omega t} \psi_{\vec{k}, \omega}(x; [\phi])$ and the self energy $\Sigma(k; [\phi]) = \Sigma(\vec{k}; [\phi])$ and renormalization factor $A_k[\phi] = A_{\vec{k}}[\phi]$ are *independent* of ω . The Green's function $G(x, x'; [\phi])$ is given by

$$G(x, x'; [\phi]) = \sum_k \frac{\psi_k(x; [\phi])\psi_k^*(x'; [\phi])}{\hbar\omega - \epsilon(\vec{k}) - \Sigma(\vec{k}; [\phi]) + i\delta_{\vec{k}}}, \quad (5)$$

and the usual perturbation expansion for G can be recovered by expanding $\psi_k[\phi]$ and $\Sigma(\vec{k}; [\phi])$ in power series of ϕ .

The advantage of introducing eigenstates ψ_k 's can be seen when we expand the Grassman fields $\Psi(x)$ in terms of ψ_k 's using the completeness relation, $\Psi(x) = \sum_k \psi_k(x; [\phi])c_k$, with an

analogous relation between $\Psi^+(x)$ and c_k^+ fields, where c_k and c_k^+ are Grassman number fields and ψ_k 's are complex-number wavefunctionals of ϕ . The time evolution operator U can be expressed in terms of path integral over c_k and c_k^+ fields through an Unitary transformation. In particular,

$$\lim_{T \rightarrow \infty} \text{Tr} [U(T, -T)] = \int Dc Dc^+ \int D\phi e^{i \sum_q \frac{\phi(q)\phi(-q)}{2v(|q|)}} e^{\frac{i}{\hbar} \int_{-T}^T dt L(\phi, c, c^+, t)}, \quad (6a)$$

where

$$L(\phi, c, c^+, t) = \sum_{\vec{k}} c_{\vec{k}}^+(t) \left[i\hbar \frac{\partial}{\partial t} - \epsilon(\vec{k}) - \Sigma(\vec{k}; [\phi]) + i\delta_{\vec{k}} \right] c_{\vec{k}}(t), \quad (6b)$$

$c_{\vec{k}}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} c_k$ and $\int Dc Dc^+ = \Pi_k \int Dc_k Dc_k^+$. Notice that although $\psi_k(x; [\phi])$'s are in general time-dependent, $\Sigma(\vec{k}; [\phi])$ is independent of t because of the independence of $\Sigma(\vec{k}; [\phi])$ on ω . In the $T \rightarrow \infty$ limit, we may integrate out *first* the ϕ field in Eq. (6b), obtaining

$$\lim_{T \rightarrow \infty} \text{Tr} U(T, -T) = \int Dc Dc^+ e^{\frac{i}{\hbar} \int_{-T}^T dt \left[\sum_{\vec{k}} c_{\vec{k}}^+(t) (i\hbar \frac{\partial}{\partial t} - \epsilon(\vec{k}) + i\delta_{\vec{k}}) c_{\vec{k}}(t) - E_{int}[n(\vec{k}, t)] \right]}, \quad (7a)$$

where

$$E_{int}[n(\vec{k}, t)] \sim (i\hbar) \ln \left[\int D\phi e^{i \sum_{\vec{q}} \frac{\phi(\vec{q}, t') \phi(-\vec{q}, t')}{2v(|\vec{q}|)}} - \frac{i}{\hbar} \sum_{\vec{k}} \Sigma(\vec{k}; [\phi]) n(\vec{k}, t) \right], \quad (7b)$$

where $n(\vec{k}, t) = c_{\vec{k}}^+(t) c_{\vec{k}}(t)$. Notice that we have extended the limit of time integration over ϕ fields from $\pm T$ to $\pm \infty$ in Eq. (7b). This is valid if T is much larger than the characteristic time scale \hbar/E_ϕ governing the fluctuations of ϕ field, where E_ϕ is the corresponding characteristic energy. Notice also that although the precise form of E_{int} is hard to determine, it is clear that E_{int} is a functional only of the occupation numbers $n(\vec{k})$. The special form of action (7a) implies that in the long-time, low energy ($\ll E_\phi$) limit, the Hamiltonian of the system, when expressed in terms of the occupation numbers $n(\vec{k})$'s, has the diagonal form

$$H = \sum_{\vec{k}} \epsilon(\vec{k}) n(\vec{k}) + E_{int}[n(\vec{k})],$$

i.e., the low energy eigenstates of the system are completely characterised by occupation numbers $n(\vec{k})$, suggesting that the states represented by $n(\vec{k})$ are in fact the quasi-particle states in Landau Fermi liquid theory [1,2].

To show that $n(\vec{k})$ indeed characterizes quasi-particle states we examine the ground state *quasi-particle* occupation number $\langle n(\vec{k}) \rangle = \langle n(\vec{k}; [\phi]) \rangle_\phi$, where

$$n(\vec{k}; [\phi]) = \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega 0^-}}{\hbar\omega - \epsilon(\vec{k}) - \Sigma(\vec{k}; [\phi]) + i\delta_{\vec{k}}},$$

is independent of ϕ , and

$$\langle F[\phi] \rangle_\phi = \frac{\int D\phi (F[\phi]) e^{iS_{eff}(\phi)}}{\int D\phi e^{iS_{eff}(\phi)}}, \quad (8)$$

where $S_{eff}(\phi) \sim \left(\sum_q \frac{\phi(q)\phi(-q)}{2v(|\vec{q}|)} - i \sum_k \ln(1 + G_0(0, k) \Sigma(\vec{k}; [\phi])) \right)$ is obtained from Eq. (6) by first integrating over the fermion fields. It is easy to see that $\langle n(\vec{k}) \rangle = \theta(-\xi_{\vec{k}})$, i.e., the quasi-particle ground state occupation number is a θ -function as the occupation number for non-interacting fermions, in agreement with Fermi liquid theory [1,2]. Notice this is a direct consequence of adiabaticity requirement, that for any configuration ϕ , the ground state is obtained by switching on ϕ adiabatically, and the occupation number $n(\vec{k}; [\phi])$ is not affected by the switching process. Notice also that in an interacting system, the quasi-particle wavefunction is a many-body wave-function. This is reflected in our formalism where ψ_k 's are functionals of auxillary ϕ field which represents density fluctuations in the system. The many-body nature of $\psi_k[\phi]$'s appears as a weighted sum over all possible configurations of ϕ (or density fluctuations) carried by quasi-particles when physical quantities are computed.

For small derivations $\delta n(\vec{k}) = n(\vec{k}) - \theta(-\xi_{\vec{k}})$, the energy of the system is given by the Landau expression,

$$E[n(\vec{k})] \sim E_0 + \sum_{\vec{k}} E_{\vec{k}} \delta n(\vec{k}) + \frac{1}{2} \sum_{\vec{k}, \vec{k}'} f_{\vec{k}\vec{k}'} \delta n(\vec{k}) \delta n(\vec{k}'), \quad (9)$$

where E_0 is the ground state energy and the quasi-particle energy dispersion $E_{\vec{k}} = \epsilon(\vec{k}) + \Sigma(\vec{k})$ and Landau parameters $f_{\vec{k}\vec{k}'}$ can be computed in our scheme as functional derivatives of E_{int} with respect to $n(\vec{k})$'s. It is straightforward to show from Eqs. (7) and (8) that $\Sigma(\vec{k}) = \delta E_{int} / \delta n(\vec{k}) = \langle \Sigma(\vec{k}; [\phi]) \rangle_\phi$ and $f_{\vec{k}\vec{k}'} = \delta \Sigma(\vec{k}) / \delta n(\vec{k}') = \langle \Sigma(\vec{k}; [\phi]) \Sigma(\vec{k}'; [\phi]) \rangle_\phi^c$, where $\langle AB \rangle_\phi^c = \langle AB \rangle_\phi - \langle A \rangle_\phi \langle B \rangle_\phi$.

The transport equation for quasi-particles can be constructed following the phenomenological approach of Landau [2] where *local* distribution of quasi-particles $n(\vec{r}, \vec{k}; t)$ is introduced and the transport equation is derived using Eq. (9) by assuming validity of usual classical kinetic theory. It has to be emphasised that the quasi-particle states which are exact eigenstates of the system are extended throughout whole space [7] and quasi-particles localized in space are only approximate eigenstates of the system and have finite life-time τ [2,7]. To derive the transport equation more rigorously we have to define first local quasi-particle states in our formulation. An effective low energy Hamiltonian for the *local* quasi-particles can then be derived to obtain the correct transport equation.

For non-interacting system a localized state can be formed by superposition of plane wave states. However, in our formulation since quasi-particle wavefunctions are functionals of ϕ which characterize density fluctuations in the system, a local quasi-particle state can be constructed only if we restrict also density fluctuations (or ϕ) carried by the quasi-particle to be local also. For *local* quasi-particle state of size scale q^{-1} , we expect that the ϕ field associated with the quasi-particle should be restricted to momentum scale $> q_c \sim q$. To implement this scheme we divide the ϕ field into slow and fast varying parts $\phi(x) = \phi_s(x) + \phi_f(x)$, where $\phi_{s(f)}(x) = \sum_{|\vec{q}| < (>) q_c} e^{i\vec{q} \cdot \vec{x}} \phi(q)$, and we shall assume that local quasi-particles of size q_c can be constructed from superposition of quasi-particle states of the corresponding interacting system, when $\phi(x)$ is replaced by $\phi_f(x)$. Notice that for a system of size $L \sim q_c^{-1}$, our local quasi-particle states become exact eigenstates of the system, as is expected physically.

With this definition of *local* quasi-particles we can derive the effective low energy Hamiltonian and transport equation for local quasi-particles straightforwardly. The algebra of this derivation is lengthy and we shall report the details in a separate paper. We find that the usual Fermi liquid transport equation is recovered in the limit $q_c \rightarrow 0$ [8].

Lastly we examine the quasi-particle renormalization factor z . To compute z , we examine the usual one-particle Green's function $g(\vec{k}, \omega)$ in our theory. Using Eq.(5), it is straightforward to show that

$$g(\vec{k}, \omega) = \sum_{k'} < \left(\frac{|\langle \psi_k^{(0)} | \psi_{k'} \rangle|^2}{\hbar\omega' - \epsilon(\vec{k}') - \Sigma(\vec{k}'; [\phi]) + i\delta_{\vec{k}'}} \right) >_{\phi}, \quad (10)$$

where $k' = (\vec{k}', \omega')$ and $\langle \psi_k^{(0)} | \psi_{k'} \rangle = \int d^{d+1}x \psi_k^{(0)*}(x) \psi_{k'}(x; [\phi])$. The bare-particle occupation number is

$$n^b(\vec{k}) = \frac{\hbar}{2\pi i} \int d\omega g(\vec{k}, \omega) = \sum_{\vec{k}'} \theta(-\xi_{\vec{k}}) < |F_{\vec{k}\vec{k}'}^{\phi}(t)|^2 >_{\phi},$$

where $F_{\vec{k}\vec{k}'}^{\phi}(t) = \int \frac{d^d x}{(2\pi)^d} e^{-i\vec{k} \cdot \vec{x}} \psi_{\vec{k}',0}(\vec{x}, t; [\phi])$. We have again made use of the result $\psi_{\vec{k}, \omega + \Omega}(x; [\phi]) = e^{-i\Omega t} \psi_{\vec{k}, \omega}(x; [\phi])$ in deriving the last expression. Assuming that the wavefunctions $\psi_k(x; [\phi])$ are smooth functions of \vec{k} which do not show any discontinuity across Fermi surface, it is easy to show using Eq. (4) that, $z = n^b(k_F^+) - n^b(k_F^-) = < |A_{k_F}[\phi]|^2 >_{\phi}$, i.e., z is equal to the weighted average over ϕ field the wavefunction overlap between non-interacting states $\psi_k^{(0)}(x)$ and $\psi_k(x; [\phi])$ on the Fermi surface, as is expected from our identification of $\psi_k(x; [\phi])$'s as quasi-particles wavefunctionals in our theory.

A few comments on our formulation of Fermi liquid theory is now in order. By organizing perturbation theory in a slightly different way, we have succeeded in formulating Fermi liquid theory directly in terms of quasi-particles in our theory. Notice that as far as an order-by-order expansion in computing quantities like one- and two- particle Green's functions are concerned, our formulation offers nothing new. However, the possibility of identifying quasi-particles in our theory enables us to compute quantities like Landau-parameters and quasi-particle energies more directly. For example, it is straightforward to show that by keeping the self-energy $\Sigma(\vec{k}; [\phi])$ to second order in ϕ , we obtain the usual RPA (Random-Phase Approximation) expression for ground state energy, with the occupation number $n^{(0)}(\vec{k})$ in the Lindhard function χ_0 replaced by the corresponding quasi-particle occupation number in our theory. The quasi-particle self-energy $\Sigma(\vec{k})$ is found to be equal to the RPA on-shell self-energy $\Sigma^{RPA}(\vec{k}, \xi_{\vec{k}})$. Notice that in general there are more than one ways of introducing the Hubbard-Strotonovich transformation and the auxillary fields, corresponding to different ways of organizing perturbation series [6]. Our results remain unchanged as long as single-particle wave-functionals $\psi_k(x; [\phi])$ can be introduced and can be obtained perturbatively

from the non-interacting states $\psi_k^{(0)}(x)$.

The main power of our formulation appears in studying Fermi liquids with $z \rightarrow 0$, where usual perturbation theory fails. Notice that $z \rightarrow 0$ does not necessarily imply breakdown of adiabaticity. For example, for a system of volume L^d , and in the presence of auxillary field ϕ , an eigenstate ψ_k is formed by mixing plane wave state $\psi_k^{(0)}$ with other plane wave states $\psi_{k'}^{(0)}$. The probability of mixing is of order $1/L^d$ for each state $\vec{k}' \neq \vec{k}$. As long as the renormalization factor $A_k[\phi]$ is of order $\geq 1/L^d$, the state ψ_k may still 'remembers' its parent plane wave state $\psi_k^{(0)}$ and adiabaticity can be kept, although z may approach zero in the $L \rightarrow \infty$ limit. Our formulation implies that a Fermi-Liquid description may still be applicable in describing the thermodynamics and low-energy, long-wavelength properties of these systems where the one-particle properties may be drastically different from usual $z \neq 0$ Fermi liquids. Examples of these systems include one-dimensional Luttinger liquids [9] where similar idea has been proposed by Carmelo *et.al.* [10] and systems with super-long-range interactions in dimensions > 1 [11]. We find that our Fermi-liquid formulation, after suitably refined, can be applied to describe these systems [8] where the one-particle properties are non-Fermi liquid like. The details of our calculations will be presented in a seperate paper.

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